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A comparison of three analytical approximations for basket option valuation

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Abstract

Three prominent analytical approximations for pricing basket options, by Levy (1992), Ju (2002) and Deelstra et al. (2004), are tested for performance and accuracy. Sensitivity analysis shows that all three have greater errors in high volatility and long maturity environments, while Deelstra has weaknesses with small correlation and baskets with few stocks. Deelstra and Levy show tendencies to underprice and overprice respectively, while Ju's errors are more consistently around the true price. A mathematical understanding of the three techniques is also developed.

Contents

1	Introduction	1
2	Review of the literature	3
3	Theoretical background	5
3.1	The Deelstra method	6
3.1.1	The concept of comonoticity	6
3.1.2	Conditioning	11
3.1.3	An exact part	13
3.1.4	Lower bound	14
3.1.5	Moment-based approximation	15
3.1.6	Choice of Λ , the conditioning variable	16
3.2	Levy's log-normal moment matching	16
3.3	Ju's Taylor expansion	17
4	Methodology and testing	20
4.1	Monte Carlo estimates	20
4.2	Analytical approximations	20
4.3	Choosing the best Deelstra	21
4.4	Testing	21
5	Results and analysis	22
5.1	Table and summary statistics	22
5.2	Sensitivity analysis	24
5.2.1	Number of stocks and correlation	24
5.2.2	Volatility and time to maturity	25
5.2.3	Basket weights and the risk free rate	26
5.2.4	Moneyness	26
6	Conclusion	27
6.1	Further research	28
	Appendices	29

A	Choosing the best Deelstra	29
A.1	Summary statistics	29
A.2	Sensitivity analysis	29
B	Some code	30
7	References	33

List of Figures

1	The sensitivity of the Levy, Ju and Deelstra approximations to N , the number of stocks, and ρ , the correlation of stocks in the basket.	24
2	The sensitivity of the Levy, Ju and Deelstra approximations to σ , the volatility, and T , the time to maturity.	25
3	The sensitivity of the Levy, Ju and Deelstra approximations to a , the basket weights, and r , the risk free rate.	26
4	The sensitivity of the Levy, Ju and Deelstra approximations to K/S , the moneyness of the option. The alternate parameters are $T = 1.5$, $r = 0.15$, $\sigma = 0.3$ while the rest stay the same.	27
5	The sensitivity of four Deelstra approximations to T , the time to maturity, and a , the basket weights.	30

List of Tables

1	A comparison of the Levy, Deelstra and Ju approximations across a range of parameter sets and compared to a Monte Carlo benchmark with given standard error.	23
2	Summary statistics for various Deelstra approximations, using the same parameter set as in table 1.	29

1 Introduction

Options are contracts that grant the right, but not the obligation, to trade equity or commodities at a predetermined price. This concept is not a new one; as far back as the fourth century BC Aristotle told of a philosopher called Thales, who made a tidy profit in the olive market with just such an agreement.

Options were popular in Amsterdam in the 17th century, when the first formalised stock market was taking shape to fund the seaborne exploration of the unknown world. They were also used by speculators in the so called 'tulip-mania' of 1634–7, which many consider the first example of that modern phenomenon, the speculative bubble.

Since then the markets have grown in sophistication and regulation. More complex financial instruments have emerged to cater for the often niche requirements of modern day corporations and investment institutions. Basket options fall into this category.

A basket option is an exotic option whose underlying is a portfolio or basket. In the case of a call, the payoff at maturity is zero if the strike is greater than the basket value, and the difference if the basket value is more than the strike. They are traded over the counter and hedged dynamically by financial institutions, usually with groups of stocks as the underlying although indices, currencies, and to a lesser extent interest rates are also possible.

Basket options are useful in a range of contexts. One of the primary uses is for investors to gain exposure to one whole industry or sector. For example, suppose an investor is confident of an upturn in the resources sector, but isn't so confident as to risk choosing one or two individual stocks. Purchasing a basket with a number of resource stocks as the underlying is a cost effective way to implement that view (Beisser 1999).

Basket options are also popular for currency hedging by multinational corporations. Often the complexities raised by exposure to multiple currencies and the correlations between them cannot be adequately handled by a mixture of forwards and vanilla calls and puts. Basket options are cheaper, and more effective hedges. A US based manufacturer who exports around the world, for instance, might buy a basket filled with long the dollar components, to hedge the

risk of dollar depreciation. The interested reader is referred to Falloon (1997) and Falloon (1998), where three well known American based companies that use basket options in this way are discussed.

Basket options can also be used by equity portfolio managers as protective puts. This is cheaper than buying many individual vanilla puts (see Smith 1998), although more expensive than buying an index option¹. So while index options are the preferable route for portfolio's benchmarked to an index, "absolute return" managers can turn to baskets to consolidate downside risk.

A final note on the usefulness of basket options. Asian options, which are essentially a special case of basket options, are popular partly because they make it much more difficult for price manipulations to affect the terminal pay-off (e.g. Lord (2006)). Baskets are the same. Potential manipulators are much less likely to attempt to influence the market, considering the large amount of capital required to turn around a whole basket of stocks.

The pricing of basket options is not trivial. When Black and Scholes (1973) derived their famous option valuation formula, it applied only in the case of vanilla calls and puts where the terminal price distribution is assumed lognormal. This is a rather restrictive assumption, and doesn't apply in the case of basket options, as the sum of lognormal stock prices is not lognormal. The pricing of exotic options such as baskets has been an area of active study for the last 30 years or more.

Pricing can take the form of many methods, but accurate analytical approximations are perhaps the most sought after in the world of exotic options; their speed is invaluable not only for pricing, but also for real-time dynamic hedging.

In this dissertation we study the effectiveness of three prominent analytical pricing methods for basket options. The first is by Levy (1992), which although older is still popular in practice. The second and third are by Deelstra et al. (2004) and Ju (2002). These use the theory of comonotonicity and perturbation theory, respectively, and are arguably the best analytical approximations available today.

This paper is structured as follows: In section 2 we review the literature concerning the pricing of basket options. Section 3 deals with the mathematical

¹Index options can be considered a special case of basket options, but they are usually exchange traded and therefore cheaper.

theory used to derive the methods, while section 4 describes the methodology used to test them. Section 5 contains the results of the tests along with some analysis, and we conclude in section 6. Two appendices contain more details.

2 Review of the literature

The methods proposed in the literature for pricing basket options can largely be placed into one of four categories: Monte Carlo simulations, tree-based methods, partial differential equations, and analytical approximations. While this paper will deal with methods in the fourth category, it is nevertheless instructive to briefly cover what else has been done.

Boyle (1977) first showed that Monte Carlo techniques, which had traditionally been used in physics applications, could also be applied to pricing financial options. His method involved simulating many random stock paths numerically, calculating the option value on each terminal stock price, and averaging over those prices to approximate the true option value. This method turned out to be extraordinarily flexible, though time consuming. Whereas the analytical solution that Black and Scholes provided could only deal with a few limiting cases, the Monte Carlo method allowed for the easy inclusion of more exotic, and perhaps realistic, parameter regimes such as volatility skews and term structures or jump processes. Lengthy calculation times could be cut down by the use of antithetic and control variables. Nevertheless, computing power has always been the determining factor in the accuracy one can achieve with Monte Carlo methods, and much work has been done to improve efficiencies, for example Boyle, Broadie and Glasserman (1997). The recent phenomenon of moving matrix intensive calculations from the CPU to the GPU has also benefitted the speed with which simulations can be done, sometimes by up to 50 or 60 times (Tomov 2005). In this dissertation, as is often the case in practice, Monte Carlo methods are used to benchmark other, less expensive, approximations.

Black and Scholes originally used arbitrage arguments to derive their famous formulation, yet it is interesting that one can arrive at exactly the same conclusion using binomial trees². The binomial tree method has been extended to price path dependant options and baskets by Hull and White (1993) and Klassen (2001), among others. Their approach considers a table of average rates at each

²The Cox Ross Rubenstein model for binomial trees leads to this result.

node in the binomial tree and with certain interpolations accurate pricing can be done using the standard backwards recursion. This method is most effective when the number of assets in the basket is small.

Other methods to price basket options have utilised the vast mathematical theory that has been developed to simplify the partial differential equations satisfied by various kinds of options. Ingersoll (1987) and Wilmott, Dewynne and Howison (1993) used a change of variables to reduce the dimension of the PDE for a floating strike Asian option, thereby vastly reducing the complexity of solving it. Rogers and Shi (1995) did the same for the fixed strike Asian option³. Zhang (2001) extends this by finding a PDE for the difference between an analytical approximation and the true price, which can then be solved numerically. His semi-analytical method reportedly achieves high accuracies at no great computational cost.

The above methods all require some form of computation, which is not always desirable in the financial world and quick, accurate analytical approximations have become increasingly important. Levy (1992) was one of the first to attempt a method of this type, "to avoid time-consuming numerical procedures". While the arithmetic sum of lognormal distributions is not lognormal, he nevertheless proposed the lognormal distribution as an approximation by matching the first two moments. This approach is popular because it allows all the prior Black-Scholes knowledge of pricing options on lognormal stock paths to be utilised as is. Ritchken et al. (1993) and Turnbull and Wakeman (1991) extend this idea by matching not only the mean and variance, but also adjusting for skewness and kurtosis, thereby taking the first four moments into account. They use the so called Edgeworth series expansion in their analysis, first introduced into the finance literature by Jarrow and Rudd (1982).

Gentle (1993) utilised the geometric average to approximate the arithmetic average, which is more common in practice. Since the geometric average of lognormal distributions is itself lognormal, standard Black-Scholes pricing follows. This method is most accurate when the weightings in the basket are equal, or close to equal.

Milevsky and Posner (1998a) used the reciprocal gamma distribution to approximate the basket, motivated by the fact that the distribution of correlated

³Since Asians are a special case of Basket options, most methods for pricing Asians can and have been applied to baskets.

lognormal random variables converges to the reciprocal gamma distribution as $n \rightarrow \infty$. Again, the first two moments are matched.

Ju (2002) extends Levy's earlier formulation by using a Taylor expansion around zero volatilities to approximate the ratio of the characteristic function of the basket to the approximating lognormal variable. This method uses the powerful techniques of perturbation theory, which are popular in other branches of applied mathematics, and the resultant approximation is among the most accurate today.

While the previous approaches rely on approximating the ungainly distribution of a basket with a more tractable distribution, Rogers and Shi (1995) and Curran (1994) take a different approach. They condition the terminal payoff of the option on a suitably chosen random variable Z using the tower property of conditional expectations and derive an integral with an analytical solution. The result is a lower bound on the price of the option which is remarkably close to the true price, although some parameter regimes show more accurate results than others. This approach has been improved upon by Beisser (1999) and Deelstra et al. (2004), the latter of which is tested in this paper.

3 Theoretical background

A basket option is an exotic option which has a payoff that depends on the value of a group of stocks, written as

$$S = \sum_{i=1}^n a_i S_i(t = T). \quad (1)$$

The positive weightings, a_i , sum to 1. An arithmetic basket call option of the European variety, to be precise, has the following value at maturity:

$$V_T = (S - K)^+, \quad (2)$$

where K is the strike determined at $t = 0$. In the risk neutral world, where discounted tradable assets have the martingale property, the value at $t = 0$ is

$$V_0 = e^{-rT} E^Q[V_T], \quad (3)$$

where Q is the risk neutral measure.

If one assumes, as Black and Scholes did famously in 1973, that stock prices follow a lognormal distribution, there is unfortunately no way of finding an analytical price for (3) at any time before maturity (except in the unlikely occurrence that the stocks are perfectly correlated or completely independant).

It is for this reason that good approximations become essential for pricing and hedging basket options, especially if one needs to do it quickly.

The methods tested here utilise non-trivial mathematical techniques and it is important to spend a little time developing their derivations. While we do not attempt an exhaustive explanation of the methods, especially of Ju, it is nevertheless instructive to understand something of their mathematical grounding.

Section 3.1 concentrates on the Deelstra method, while sections 3.2 and 3.3 deal with the Levy and Ju methods respectively.

3.1 The Deelstra method

This method comes out of the body of work begun by the academic actuaries at Leuven, Belgium, on the concept of comonotonicity and its applications in finance. In the next section we lay some groundwork for understanding comonotonicity, which is integral to the Deelstra method.

The following section is based on the research report of Dhaene et al. (2002a), which is doubtless the introduction for many a newcomer to the field.

3.1.1 The concept of comonotonicity

Comonotonicity has been developed in the last 15 to 20 years as a very useful tool for approximating sums of random variables when the marginal distributions are known but the dependance structure is not. In our case, the random variables being summed are the weighted terminal values of the stocks or indices which make up the basket.

If one assumes mutual independence for the individual terms in the sum, the mathematics for valuation becomes quite tractable and there is, in fact, a closed form solution for (3). Realistically however, stocks or indices are almost never independant. One needs to take account of the dependance structure when pricing basket options.

Consider X , the sum of random variables for which the marginals are known but the joint distribution is not. The method of comonotonicity finds another random variable Y , such that it is always “less attractive” to pay Y than to pay X . It is in effect a sort of upper bound. To quantify this concept we need some definitions.

Definition 1 *The stop-loss premium for a random variable X is $E[(X - d)^+]$, for $d \in \mathbb{R}$.*

This is defined in the actuarial sense, but is clearly analogous to the terminal value of a standard call option in finance.

It can be shown using integration by parts that

$$E[(X - d)^+] = \int_d^\infty (1 - F_X(x))dx, \quad \forall d \in \mathbb{R},$$

where $F_X(x)$ is the distribution function of X . Thus the stop-loss premium can be considered as the weight of the upper tail of the distribution function of X .

Now we can begin to *order* random variables in the following way

Definition 2 *Consider two random variables, X and Y . X is said to precede Y in the stop-loss order sense, iff*

$$E[(X - d)^+] \leq E[(Y - d)^+], \quad \forall d \in \mathbb{R}.$$

This is denoted $X \leq_{sl} Y$.

Therefore, if X precedes Y in the stop-loss order sense, X has uniformly smaller upper tails than Y , and a payment of X is indeed more attractive than a payment of Y as it is less risky. One might think this result strange, as it says nothing of the means of the distributions, but it turns out that $X \leq_{sl} Y \implies E[X] \leq E[Y]$.⁴

In our pursuit of finding a random variable Y which is less attractive than X , we would naturally prefer Y to approximate X as closely as possible. Thus we choose the case where the expectations are the same: $E[X] = E[Y]$. This case leads to a new type of order defined as follows:

⁴Lack of space precludes inclusion of the proof/proofs here, but the interested reader is referred to Dhaene et al. (2002a) for more details.

Definition 3 Consider two random variables, X and Y . X is said to precede Y in the convex order sense iff

1. $E[(X - d)^+] \leq E[(Y - d)^+], \quad \forall d \in \mathbb{R}$
2. $E[X] = E[Y]$

This is denoted as $X \leq_{cx} Y$.

Convex order turns out to be a more powerful concept than stop-loss order, and we will use it to order random variables for the remainder of this paper.

Whereas stop-loss precedence implied lighter upper tails, convex order implies both lighter upper tails and lighter lower tails⁵.

$$X \leq_{cx} Y \implies \begin{cases} E[(X - d)^+] \leq E[(Y - d)^+], & \forall d \in \mathbb{R} \\ E[(d - X)^+] \leq E[(d - Y)^+], & \forall d \in \mathbb{R} \end{cases}$$

This is an important development when we describe X as a “more attractive” payment than Y - while there is less upside risk, as before, there is also less downside risk.

The following are a number of important results concerning convex order²:

Proposition 1 Let X and Y be two random variables. Then

1. $X \leq_{cx} Y \iff -X \leq_{cx} -Y$
2. $X \leq_{cx} Y \implies \text{Var}[X] \leq \text{Var}[Y]$
3. $(X \leq_{cx} Y \wedge \text{Var}[X] = \text{Var}[Y]) \iff X \stackrel{d}{=} Y$

1.1 shows that in terms of convex order, the interpretation of X and Y as payments or gains is irrelevant - what matters are the extreme values, whether in the upper tail or in the lower tail.

1.2 shows again how X is less risky⁶ than Y . The reverse implication is not true in general.

1.3: If the variances are equal, then convex order implies that X and Y are equal in distribution.

⁵Lower tails because $E[(d - X)^+] = \int_{-\infty}^d F_X(x)dx$

⁶Here we assume the early convention adopted by Markowitz (1959) and others, that high variance implies high risk.

Now that we have defined an order on random variables which relates their “riskiness”, we go about finding a random variable Y to approximate the sum X .

The question goes as follows. Consider $S = \sum X_i$, a sum of random variables; the marginals of each X_i are known, but the joint distribution is not. Can we impose a joint distribution on the X_i ’s such that the resultant $S^c = \sum X_i^c$ is always larger in a convex order sense than any other possible S ?

The answer is yes, and this joint distribution is called the comonotonic distribution.

First we need to define the concept of componentwise ordering:

Definition 4 Two vectors $\mathbf{X} = (x_1, x_2, \dots, x_n)$ and $\mathbf{Y} = (y_1, y_2, \dots, y_n)$ are said to be ordered componentwise iff

$$\begin{aligned} x_1 \leq y_1 &\implies x_i \leq y_i \quad \forall i \\ \text{and } x_1 \geq y_1 &\implies x_i \geq y_i \quad \forall i. \end{aligned}$$

We now broaden this concept to random vectors and define comonotonicity in the process:

Definition 5 A random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is said to be comonotonic iff any two outcomes are ordered componentwise, a.s.

There are two additional necessary and sufficient conditions for comonotonicity which are very useful.

Theorem 1 A random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is comonotonic iff one of the following equivalent conditions hold:

1. For all $\mathbf{x} = (x_1, x_2, \dots, x_n)$, we have

$$F_{\mathbf{X}}(\mathbf{x}) = \min\{F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n)\}. \quad (4)$$

2. For $U \sim \text{Uniform}(0, 1)$, we have

$$\mathbf{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \dots, F_{X_n}^{-1}(U)). \quad (5)$$

As before, we leave the proof to Dhaene et al. (2002a).

Equation 5 in the theorem is most instructive for understanding how the comonotonic distribution is constructed⁷. The well known probability integral transform says that for a continuous random variable X and its distribution function F_X we have the following:

$$F_X(X) = U \implies X = F_X^{-1}(U),$$

where $U \sim \text{Uniform}(0, 1)$.

The key to understanding (5) is that \mathbf{X} is made up of inverse transforms of the *same* uniform random variable. So an instance of \mathbf{X} could be $(F_{X_1}^{-1}(0.38), F_{X_2}^{-1}(0.38), \dots, F_{X_n}^{-1}(0.38))$, for example. And because all the $F_{X_i}^{-1}$'s are increasing, all the resulting instances of \mathbf{X} are necessarily ordered componentwise.

Equations 4 and 5 also show that one only needs the marginals to construct the comonotonic joint distribution, which is a key prerequisite for what follows.

Now that we have laid the foundation of comonotonicity, we come to perhaps the most useful result in this section.

Theorem 2 *Consider a random variable $\mathbf{X} = (X_1, X_2, \dots, X_n)$ and its sum $S = \sum X_i$, where the marginals are known but not the joint distribution. Let \mathbf{X}^c be the comonotonic counterpart to \mathbf{X} , and $S^c = \sum X_i^c$. Then*

$$S \leq_{cx} S^c.$$

This can be also be put as follows:

$$X_1 + X_2 + \dots + X_n \leq_{cx} F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \dots + F_{X_n}^{-1}(U). \quad (6)$$

Note that this result does not depend on the existing joint distribution in S . S^c is the least desirable, most risky, outcome in every case. Thus it can be considered an upper bound.

The next important question is whether it is possible to find the stop-loss premium for S^c . If not, it is not a useful approximation or upper bound for S

⁷In the following analysis we will deal only with continuous random variables which have strictly increasing distribution functions.

but only a theoretical result. As it turns out, things pan out rather nicely:

Theorem 3 *Consider $S^c = \sum X_i^c$, where X_i^c is the comonotonic counterpart for X_i for all i . Then*

$$E[(S^c - d)^+] = \sum E[(X_i - F_{X_i}^{-1}(F_{S^c}(d)))^+], \quad (7)$$

where

$$F_{S^c}(d) = \sup \left\{ d \in (0, 1) \mid \sum F_{X_i}^{-1}(d) \leq x \right\}. \quad (8)$$

This shows that finding the stop-loss premium for S^c reduces to finding stop-loss premia for the individual random variables X_i .

3.1.2 Conditioning

Rogers and Shi (1995) and Kaas et al. (2000) showed that it is possible to use conditioning variables to find and improve bounds on Asian and basket type options. This can be combined with comonotonicity, as the following section elaborates.

Upper bounds Let us assume that we have further information about the dependance structure of $\mathbf{X} = (X_1, X_2, \dots, X_n)$ that is contained in some random variable Z . We assume that Z is a function of \mathbf{X} and that we know its distribution, as well as the conditional distribution of each $X_i|Z$.

We now create the comonotonic distribution for $\mathbf{X}|Z = (X_1|Z, X_2|Z, \dots, X_n|Z)$ and introduce the notation $F_{X_i|Z}^{-1}(U)$. This is the inverse distribution of $X_i|Z$ with the usual uniform random variable for comonotonicity. It turns out that a sum of such conditioned comonotonic variables is also an upper bound for S , as in (6).

Theorem 4 *Let $U \sim \text{Uniform}(0, 1)$, and consider a random variable Z which is independant of U . Then*

$$X_1 + X_2 + \dots + X_n \leq_{cx} F_{X_1|Z}^{-1}(U) + F_{X_2|Z}^{-1}(U) + \dots + F_{X_n|Z}^{-1}(U) = S'^c. \quad (9)$$

We leave the proof to Kaas et al. (2000).

In view of Theorem 2, we have that the conditioned comonotonic sum preceeds the usual comonotonic sum in convex order and is thus a closer approxi-

mation.

$$F_{X_1|Z}^{-1}(U) + F_{X_2|Z}^{-1}(U) + \dots + F_{X_n|Z}^{-1}(U) = S'^c \leq_{cx} F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \dots + F_{X_n}^{-1}(U) = S^c. \quad (10)$$

It should be said that the choice of Z matters a great deal in the amount of improvement that equation (10) offers. If Z is independant of \mathbf{X} , then $S'^c = S^c$, which is of no use. Choosing the optimal conditioning variable is an important part of bringing the approximation as close as possible to the true value (see section 3.1.6).

Lower bounds It is also possible to get a lower bound in terms of convex order with a conditioning variable.

Proposition 2 *For any random vector \mathbf{X} and random variable Z , we have*

$$S^l \equiv E[X_1|Z] + E[X_2|Z] + \dots + E[X_n|Z] \leq_{cx} X_1 + X_2 + \dots + X_n. \quad (11)$$

The proof employs the theorem of iterated expectations and Jensen's inequality. In effect, by conditioning upon Z , we easily find a lower bound in the convex order sense, which is very useful. Further, if the terms comprising S^l are comonotonic, we can use Theorem 3 to express their values in terms of stop-loss premiums.

If we assume that the random variable Z is such that all $E[X_i|Z]$ are non-decreasing and continuous functions of Z , as well as that the distribution functions of $E[X_i|Z]$ are strictly increasing and continuous, then the distribution function of S^l is also strictly increasing and continuous. Using Theorem 3, the stop-loss premiums of S^l are:

$$E[(S^l - d)^+] = \sum_{i=1}^n E\left[\{E[X_i|Z] - E[X_i|Z = F_Z^{-1}(F_{S^l}(d))]\}^+\right]. \quad (12)$$

To summarise the above two sections: we have lower and upper bounds such that

$$S^l \leq_{cx} S \leq_{cx} S^c. \quad (13)$$

Having developed some of the important theory considering comonotonicity and conditioning, we move on to how Deelstra et al. (2004) used these concepts in their derivations.

First, they noticed that there is a part of the basket price that can be calculated in an exact way. The remaining part they approximate with lower and upper bounds and by moment based approximations.

3.1.3 An exact part

We choose a normally distributed random variable Λ such that $\exists d_\Lambda \in \mathbb{R}$ for which $\Lambda \geq d_\Lambda \implies S \geq K$. For the moment we will assume that such a Λ exists, and later we will demonstrate with examples. This Λ will be our conditioning variable.

For such Λ we can decompose the option price into two parts - one which can be calculated exactly and one which will be approximated. Deelstra et al. (2004) showed with Monte Carlo numerics that the exact part makes up more than 90% of the full price.

From (3):

$$\begin{aligned} e^{-rT} E^Q[(S - K)^+] &= e^{-rT} E^Q[E^Q[(S - K)^+ | \Lambda]] \\ &= e^{-rT} \left\{ \int_{-\infty}^{d_\Lambda} E^Q[(S - K)^+ | \Lambda = \lambda] f_\Lambda(\lambda) d\lambda + \int_{d_\Lambda}^{+\infty} E^Q[S - K | \Lambda = \lambda] f_\Lambda(\lambda) d\lambda \right\} \end{aligned} \quad (14)$$

The first line makes use of the law of iterated expectations, while the second is just the expectation integral split into two parts.

We now assume that each stock in the basket, S_i , is lognormally distributed as follows:

$$S_i(t) = \alpha_i(t) e^{Y_i(t)}. \quad (15)$$

where $\alpha_i(t) = a_i S_i(0) e^{(r - \sigma_i^2/2)t}$ and $Y_i(t) = \sigma_i W_i(t) \sim N(0, \sigma_i^2 t)$. If (Y_i, Λ) is bivariate normally distributed for all i , with $r_i = \text{cov}^Q(Y_i, \Lambda)$, then the second part of equation (14) can be expressed exactly as follows:

$$e^{-rT} \int_{d_\Lambda}^{+\infty} E^Q[S - K | \Lambda = \lambda] f_\Lambda(\lambda) d\lambda = \sum_{i=1}^n a_i S_i(0) \Phi(r_i \sigma_i \sqrt{T} - d_\Lambda^*) - e^{-rT} K \Phi(-d_\Lambda^*). \quad (16)$$

Here Φ is the cumulative standard normal function and $d_{\Lambda}^* = \frac{d_{\Lambda} - E^Q[\Lambda]}{\sigma_{\Lambda}}$. The derivation of (16) can be found in Deelstra et al. (2004).

3.1.4 Lower bound

Using Jensen's inequality the first term in (14) can be bounded below as follows:

$$\int_{-\infty}^{d_{\Lambda}} E^Q[(S - K)^+ | \Lambda = \lambda] f_{\Lambda}(\lambda) d\lambda \geq \int_{-\infty}^{d_{\Lambda}} (E^Q[S | \Lambda = \lambda] - K)^+ f_{\Lambda}(\lambda) d\lambda. \quad (17)$$

We adopt the following notation, as in (11):

$$S^I = E[S | \Lambda]. \quad (18)$$

By adding the exact part in (14), we have the inequality

$$E^Q[(S - K)^+] \geq E^Q[(S^I - K)^+]. \quad (19)$$

Using the results from sections 3.1.1 and 3.1.2 and especially equation (12), along with the famous Black Scholes formula for option prices, we obtain the following lower bound for the price of a basket call option:

$$V_0 \geq \sum_{i=1}^n a_i S_i(0) \Phi \left[\sigma_i \sqrt{T} r_i - \Phi^{-1}(F_{S^I}(K)) \right] - e^{-rT} K (1 - F_{S^I}(K)). \quad (20)$$

This holds for any positive K and where $F_{S^I}(K)$ solves

$$\sum_{i=1}^n a_i S_i(0) e^{(r - r_i^2 \sigma_i^2 / 2)T + r_i \sigma_i \sqrt{T} \Phi^{-1}(F_{S^I}(K))} = K. \quad (21)$$

(21) requires an optimisation routine to be solved.

This lower bound can be written as a weighted average of Black and Scholes prices with modified underlying stocks, volatilities and strikes. The new stocks are \bar{S}_i with $\bar{S}_i(0) = S_i(0)$. The new volatilities for these stocks are $\bar{\sigma}_i = \sigma_i r_i$ and the strikes \bar{K}_i , $i = 1, \dots, n$ are given by

$$\bar{K}_i = \bar{S}_i(0) e^{(r - \bar{\sigma}_i^2 / 2)T + \bar{\sigma}_i \sqrt{T} \Phi^{-1}(F_{S^I}(K))}. \quad (22)$$

The lower bound in terms of these new Black-Scholes prices is then:

$$V_0 \geq \sum_{i=1}^n a_i [\bar{S}_i(0) \Phi(d_{1i}) - e^{-rT} \bar{K}_i \Phi(d_{2i})], \quad (23)$$

where

$$d_{1i} = \frac{\ln(\bar{S}_i(0)/\bar{K}_i) + (r + \sigma_i^2/2)T}{\bar{\sigma}_i \sqrt{T}} \quad \text{and} \quad d_{2i} = d_{1i} - \bar{\sigma}_i \sqrt{T}, \quad i = 1, \dots, n.$$

Deelstra et al. (2004) also derive an upper bound using similar techniques, but its performance has been shown to be far inferior to the lower bound, so we will not discuss it here.

3.1.5 Moment-based approximation

This method uses moment matching to approximate the inexact part of (14). $\mathbb{S}|\Lambda = \lambda$ is a sum of n lognormal variables. We approximate this sum, assuming it is lognormally distributed, by matching the first two moments in the standard way (see section 3.2), remembering to include the conditioning variable Λ . This leads to an approximation for the inexact part which is just a Black Scholes price on the derived first and second moments:

$$\int_{-\infty}^{d_\Lambda} E^Q[(S - K)_+ | \Lambda = \lambda] dF_\Lambda(\lambda) \approx \int_{-\infty}^{d_\Lambda} [E^Q[\mathbb{S} | \Lambda = \lambda] \Phi(d_1(\lambda)) - K \Phi(d_2(\lambda))] dF_\Lambda(\lambda), \quad (24)$$

with

$$d_1(\lambda) = \frac{(\ln E^Q[\mathbb{S}^2 | \Lambda = \lambda]) / 2 - \ln(K)}{\sigma}, \quad d_2(\lambda) = d_1(\lambda) - \sigma,$$

where

$$\sigma^2 = \ln E^Q[\mathbb{S}^2 | \Lambda = \lambda] - 2 \ln E^Q[\mathbb{S} | \Lambda = \lambda]$$

is the matched volatility.

3.1.6 Choice of Λ , the conditioning variable

It is important that the conditioning variable used contains as much information about the basket, \mathbb{S} , as possible. The three candidates proposed by Deelstra et al. (2004) are as follows.

$$\begin{aligned} \text{FA1} &= \sum_{i=1}^n e^{(r-\sigma_i^2/2)T} a_i S_i(0) \sigma_i W_i(T); \\ \text{FA2} &= \sum_{i=1}^n a_i S_i(0) \sigma_i W_i(T); \\ \text{GA} &= \frac{\ln \mathbb{G} - E^Q[\ln \mathbb{G}]}{\sqrt{\text{var}^Q[\ln \mathbb{G}]}} = \frac{\sum_{i=1}^n a_i \sigma_i W_i(T)}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j \rho_{ij} T}}. \end{aligned}$$

FA1 and FA2 are simple linear transformations of \mathbb{S} , while GA is the standardised logarithm of the geometric average \mathbb{G} , where

$$\mathbb{G} = \prod_{i=1}^n S_i(T)^{a_i}.$$

Tests (in the course of this dissertation and by Deelstra et al. (2004)) show that FA2 is the optimal conditioning variable for pricing baskets.

3.2 Levy's log-normal moment matching

Consider a N -asset market with usual lognormal stock paths as follows:

$$S_i(t) = S_i e^{(g_i - \sigma_i^2/2)t + \sigma_i w_i(t)}, \quad i = 1, 2, \dots, N, \quad (25)$$

where $g_i = r - \delta_i$ is the stock drift corrected for continuous dividend yield and σ_i and $w_i(t)$ are the i th stock's volatility and Brownian motion respectively. The w_i 's are correlated as follows $\text{corr}(w_i(t), w_j(t)) = \rho_{ij}$.

We define the basket as:

$$\mathbb{S}(t) = \sum_{i=1}^N a_i S_i(t), \quad (26)$$

where a_i are the stock weights.

Although $\mathbb{S}(T)$ is not lognormal, Levy approximates it with a lognormal

distribution, e^X , with mean M and variance $M^2 - V^2$. Then

$$\begin{aligned} E[S(T)] &= \sum_{i=1}^n a_i S_i e^{g_i T} \equiv M \quad \text{and} \\ E[S^2(T)] &= \sum_{i=1}^n \sum_{j=1}^n a_i S_i e^{g_i T} a_j S_j e^{g_j T} e^{\sigma_i \sigma_j \rho_{ij} T} \equiv V^2. \end{aligned}$$

If $X \sim N(m, v^2)$, then we also have

$$\begin{aligned} E[S(T)] &\approx E[e^X] = e^{m + \frac{1}{2}v^2} \quad \text{and} \\ E[S^2(T)] &\approx E[e^{2X}] = e^{2m + 2v^2}. \end{aligned}$$

Solving simultaneously we get

$$m = 2 \log(M) - \frac{1}{2} \log(V^2) \quad \text{and} \quad (27)$$

$$v^2 = \log(V^2) - 2 \log(M). \quad (28)$$

Now we can use the Black-Scholes formula for e^X , having matched the mean and variance of X with the first two moments of the basket S . We are left with the following approximation:

$$V_0 = e^{-rT} (M \Phi(d_1) - K \Phi(d_2)), \quad (29)$$

where Φ is the standard normal cumulative distribution and

$$d_1 = \frac{m - \ln(K) + v^2}{v}, \quad \text{and} \quad d_2 = d_1 - v.$$

3.3 Ju's Taylor expansion

Consider a N -asset market as in equation 25 above.

We seek to use a Taylor expansion around zero volatilities. This might seem implausible, as the volatilities are different for each stock, but we can overcome this by introducing a scaling parameter z .

Consider a fictitious market as follows:

$$S_i(z, t) = S_i e^{(r - z^2 \sigma_i^2 / 2)t + z \sigma_i w_i(t)} \quad i = 1, 2, \dots, N.$$

Here z scales each volatility and when $z = 1$ we recover the original stock

paths.

We define the value of the basket as in (1), with z included:

$$\mathbb{S}(z) = \sum_{i=1}^N a_i S_i(z, T),$$

where a_i are the stock weights as before. For a european call, the terminal value of a basket option with strike K is then:

$$V_T = (\mathbb{S}(1) - K)^+.$$

For simplicity define $\bar{S}_i = a_i S_i e^{\theta_i T}$ and $\bar{\rho}_{ij} = \rho_{ij} \sigma_i \sigma_j T$. The mean and variance of $\mathbb{S}(z)$ are then as follows:

$$U_1 = \sum_{i=1}^N \bar{S}_i = \mathbb{S}(0) \quad (30)$$

$$U_2(z^2) = \sum_{i=1}^N \sum_{j=1}^N \bar{S}_i \bar{S}_j e^{z^2 \bar{\rho}_{ij}} \quad (31)$$

As Levy did, we now match the first two moments of $\mathbb{S}(z)$ with a lognormal variable, but also including the scaling parameter z . Let $e^{Y(z)}$ be a lognormal random variable, with $Y(z)$ normal. Then the mean, $m(z^2)$, and variance, $v(z^2)$, of $Y(z)$ are as follows:

$$m(z^2) = 2 \log U_1 - \frac{1}{2} \log U_2(z^2) \quad (32)$$

$$v(z^2) = \log U_2(z^2) - 2 \log U_1 \quad (33)$$

We now find the density function of $X(z)$, where $X(z) = \log \mathbb{S}(z)$. To do this we consider its characteristic function,

$$E[e^{i\phi X(z)}] = E[e^{i\phi Y(z)}] \frac{E[e^{i\phi X(z)}]}{E[e^{i\phi Y(z)}]} = E[e^{i\phi Y(z)}] f(z), \quad (34)$$

where

$$E[e^{i\phi Y(z)}] = e^{i\phi m(z^2) - \phi^2 v(z^2)/2}$$

is the characteristic function of the normal random variable $Y(z)$ and

$$f(z) = \frac{E[e^{i\phi X(z)}]}{E[e^{i\phi Y(z)}]} = E[e^{i\phi X(z)}] e^{-(i\phi m(z^2) - \phi^2 v(z^2)/2)}$$

is the ratio of the characteristic function of $X(z)$, which represents the basket, to that of $Y(z)$, which represents the approximation. It is on this ratio $f(z)$ that we perform a Taylor expansion around $z = 0$ up to z^6 , which leads to

$$f(z) \approx 1 - i\phi d_1(z) - \phi^2 d_2(z) + i\phi^3 d_3(z) + \phi^4 d_4(z),$$

where $d_i(z)$ are polynomials of z and terms of higher order than z^6 are ignored.

Finally, $E[e^{i\phi X(1)}]$ is approximated as follows:

$$E[e^{i\phi X(1)}] \approx e^{i\phi m(1) - \phi^2 v(1)/2} (1 - i\phi d_1(z) - \phi^2 d_2(z) + i\phi^3 d_3(z) + \phi^4 d_4(z)). \quad (35)$$

We then find the density function of $X(1)$ by integrating this approximation over the real line and multiplying by $\frac{1}{\sqrt{2\pi}}$. We have

$$X(1) \approx h(x) = p(x) + \left(d_1(1) \frac{d}{dx} + d_2(1) \frac{d^2}{dx^2} + d_3(1) \frac{d^3}{dx^3} + d_4(1) \frac{d^4}{dx^4} \right) p(x), \quad (36)$$

where $p(x)$ is the normal density with mean $m(1)$ and variance $v(1)$.

The price of the basket call is then given by

$$\begin{aligned} V_0 &= e^{-rT} E[e^{X(1)} - K]^+ \\ &= \left[U_1 e^{-rT} \Phi(y_1) - K e^{-rT} \Phi(y_2) \right] + \left[e^{-rT} K (z_1 p(y) + z_2 \frac{dp(y)}{dy} + z_3 \frac{d^2 p(y)}{dy^2}) \right] \end{aligned} \quad (37)$$

where

$$y = \log(K), \quad y_1 = \frac{m(1) - y}{\sqrt{v(1)}} + \sqrt{v(1)}, \quad y_2 = \frac{m(1) - y}{\sqrt{v(1)}},$$

and

$$z_1 = d_2(1) - d_3(1) + d_4(1), \quad z_2 = d_3(1) - d_4(1), \quad z_3 = d_4(1).$$

Note that the terms in the first pair brackets in the sum are Levy's approximation, while the terms in the second pair of brackets are Ju's higher order corrections.

4 Methodology and testing

To compare the Deelstra, Ju and Levy approximations outlined above, we specify a comprehensive range of parameter values and test the accuracy of the approximations in each case using Monte Carlo estimates as the benchmark.

4.1 Monte Carlo estimates

As mentioned in section 2, the Monte Carlo method is often used to provide baseline valuations for exotic options. This is because it is possible to get arbitrarily close to the true option value by steadily increasing the number of iterations, though this is dependent on computing power and the amount of time available. For our Monte Carlo runs we simulate 10^{10} possible baskets for each option, which is more than enough to get accurate results and in most cases the standard error is of the order 10^{-4} or less. The matrix intensive operations are done on the GPU which is quicker than the CPU; general runnings times are under 30 minutes per option, depending on the number of stocks.

Basket options are not path dependent, so sophisticated random numbers like mersenne twisters aren't as important and the standard random number generator in Matlab proves to be adequate - there is no bias when comparing our option values to their counterparts in Ju (2002) and Deelstra et al. (2004)⁸.

4.2 Analytical approximations

We use the formulations developed in section 3 to code the three methods to be tested. Matlab is used throughout. It should be noted that the Deelstra approximation does require some numerical computation, although slight. Equation 33 of their paper (and equation 21 in this paper) has no analytical solution, and requires a non-linear solver.

⁸Bias in Monte Carlo pricing as a result of inadequate random numbers is more of a problem with path dependant options, where irregularities can 'stack up'; this is not the case with basket options, which are path independent.

4.3 Choosing the best Deelstra

In Deelstra et al. (2004) the mathematical techniques used yield a whole range of approximations. This is first of all due to the conditioning variable used, of which they propose three. These are denoted FA1, FA2 and GA. The two FA variables are linear transformations of first order basket approximations, while GA is a logarithm of the geometric average. Further, the comonotonic approach yields an upper bound, a lower bound and the moment matching middle, of which there are two types, \mathbb{S} and \mathbb{H} . The \mathbb{H} type can be done with fixed moments or variable moments.

Altogether this accounts for 15 distinct formulations for the analytical approximation. Deelstra et al. do test these variations against Monte Carlo benchmarks in section 7 of their paper and suggest that moment matching with the conditioning variable FA2 is the best formulation. We have found the same preference in the course of testing (see Appendix A for more detail), and use this formulation to represent the Deelstra method in this paper.

4.4 Testing

Two types of test are presented in this paper. First is a table of analytical valuations compared to Monte Carlo benchmarks in the tradition of the literature; second is sensitivity analysis.

For both we use the same base basket, which is an equally weighted two stock portfolio with a maturity of 1 year, $S_0 = 100$, a strike of $K = 110$, a risk free rate of 0.05, a volatility of $\sigma = 0.2$ for both stocks, and the following correlation matrix:

$$\begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}$$

For the ‘table and summary statistic’ approach in section 5.1 24 parameter sets with variations on the base parameters are chosen to expose the approximations to a range of scenarios in which to test them.

In section 5.2 sensitivity analysis is carried out for a range of each of the seven parameters, starting from the base set.

The parameters are varied one at a time as follows:

- Number of stocks (N) : 2 to 10, increments of 1
- Volatility (σ) : 0.01 to 0.80, increments of 0.01
- Time to maturity (T) : 0.5 to 5 years, increments of 0.5
- Risk free rate (r) : 0.01 to 0.5, increments of 0.01
- Correlation (ρ) : 0.02 to 0.8,⁹ increments of 0.02
- Moneyness (K/S) : 0.8 to 1.3, increments of 0.02
- Stock weights (a) : (0.01, 0.99) to (0.5, 0.5), increments of 0.01

To improve the smoothness of the sensitivity analysis graphs we run 5 sets of Monte Carlo estimates for each parameter range. This nesting also reduces the standard error by a factor of $1/\sqrt{5}$.

5 Results and analysis

5.1 Table and summary statistics

Table 1 shows prices generated by the Levy, Deelstra and Ju formulations for a range of parameter sets, as well as the Monte Carlo benchmark.

What is perhaps most striking when first looking at the results is the remarkable accuracy achieved by all three of the methods in most cases. Almost all of the time they are accurate to within 10^{-2} of the Monte Carlo price. This is good enough for use in practice, and says much for the quality of approximations possible in the Black Scholes world of tricky lognormal prices.

Secondly, the discrepancy from the benchmark generally increases as the standard error of the Monte Carlo approximation increases. This is for certain predictable parameter ranges, such as high volatility or time to maturity. It means that in practice these ranges need to be approached with caution, as not only are the analytical approximations slightly off kilter, but the Monte Carlo valuations need to be run at higher iterations to achieve the same level of accuracy.

In terms of the summary statistics, Levy clearly fares the worst in both RMSE (root mean square error) and MAE (maximum absolute error), as was

⁹These are the off-diagonal components in the correlation matrix.

Table 1: A comparison of the Levy, Deelstra and Ju approximations across a range of parameter sets and compared to a Monte Carlo benchmark with given standard error.

Parameters			Levy	Deelstra	Ju	MC	s.e. ($\times 10^{-4}$)
$\sigma = 0.20$	$N = 2$	$\rho = 0.3$	4.5262	4.5257	4.5263	4.5262	0.88
		$\rho = 0.6$	5.2101	5.2100	5.2101	5.2101	1.01
	$N = 8$	$\rho = 0.3$	3.0998	3.0998	3.0998	3.0998	0.63
		$\rho = 0.6$	4.5161	4.5161	4.5161	4.5161	0.88
$\sigma = 0.45$	$N = 2$	$\rho = 0.3$	12.6612	12.6459	12.6537	12.6529	2.53
		$\rho = 0.6$	14.1437	14.1397	14.1424	14.1421	2.87
	$N = 8$	$\rho = 0.3$	9.3079	9.3075	9.3071	9.3070	1.80
		$\rho = 0.6$	12.5442	12.5441	12.5441	12.5445	2.50
$K/S = 0.8$	$\sigma = 0.20$	$N = 2$	24.1646	24.1646	24.1643	24.1643	1.58
		$N = 8$	23.9561	23.9561	23.9561	23.9561	1.24
	$\sigma = 0.45$	$N = 2$	28.0822	28.0770	28.0674	28.0696	3.36
		$N = 8$	26.0664	26.0663	26.0640	26.0638	2.62
$K/S = 1.3$	$\sigma = 0.20$	$N = 2$	0.7911	0.7911	0.7915	0.7915	0.37
		$N = 8$	0.2511	0.2511	0.2511	0.2511	0.17
	$\sigma = 0.45$	$N = 2$	7.0344	7.0317	7.0349	7.0331	1.96
		$N = 8$	4.0730	4.0729	4.0739	4.0740	1.24
$a = 0.3$	$\rho = 0.3$	$\sigma = 0.20$	4.7918	4.7909	4.7911	4.7913	0.94
		$\sigma = 0.45$	13.2607	13.2011	13.2051	13.2082	2.68
	$\rho = 0.6$	$\sigma = 0.20$	5.3494	5.3491	5.2492	5.3492	1.04
		$\sigma = 0.45$	14.4584	14.4416	14.4432	14.4435	2.95
$T = 3$	$\sigma = 0.2$	$r = 0.05$	13.7358	13.7297	13.7322	13.7321	2.11
		$r = 0.15$	31.0114	31.0086	31.0074	31.0081	2.71
	$\sigma = 0.45$	$r = 0.05$	27.4026	27.3185	27.2541	27.2855	5.83
		$r = 0.15$	39.4996	39.4288	39.3406	39.3849	6.49
RMSE			0.0354	0.0115	0.0111		
MAE			0.1171	0.0439	0.0443		

Where left unspecified, the standard parameters are $\sigma = 0.2$, $N = 2$, $a = 0.5$, $\rho = 0.3$, $K/S = 1.1$, $T = 1$ and $r = 0.05$, where $a = 0.5$ indicates equal weights in the basket. RMSE is root mean square error, MAE is maximum absolute error.

expected. The more recent Deelstra and Ju approximations perform comparably well, with very similar RMSE's. The maximum errors for both of them occur in the doubly difficult region of high volatility and long maturities¹⁰, and are definite outliers. If one ignores this one result the MAE's are much lower.

¹⁰In the Black Scholes model, volatility and time to maturity can be considered interchangeable as they appear next to each other in the formula, thus having the same type of effect on approximation stability.

5.2 Sensitivity analysis

Sensitivity analysis is a very useful tool for presenting the performance of analytical approximations, as opposed to the more traditional ‘table and summary statistic’ format used in section 5.1. While we were unable to find a preference between the Deelstra and Ju methods with the 24 parameter sets above, sensitivity analysis provides a much deeper insight into their performance.

With the following graphs we examine how the three approximations in this study perform as all seven of the input parameters are varied and pinpoint trends and weaknesses.

The graphs show the the approximations minus the ‘true’ Monte Carlo price on the y-axis, with the relevant parameter range on the x-axis. The dotted lines are the error bars of the Monte Carlo valuation, i.e. if one of the analytical valuations falls within the error bars, then the two prices are for all intents and purposes indistinguishable at that point.

5.2.1 Number of stocks and correlation

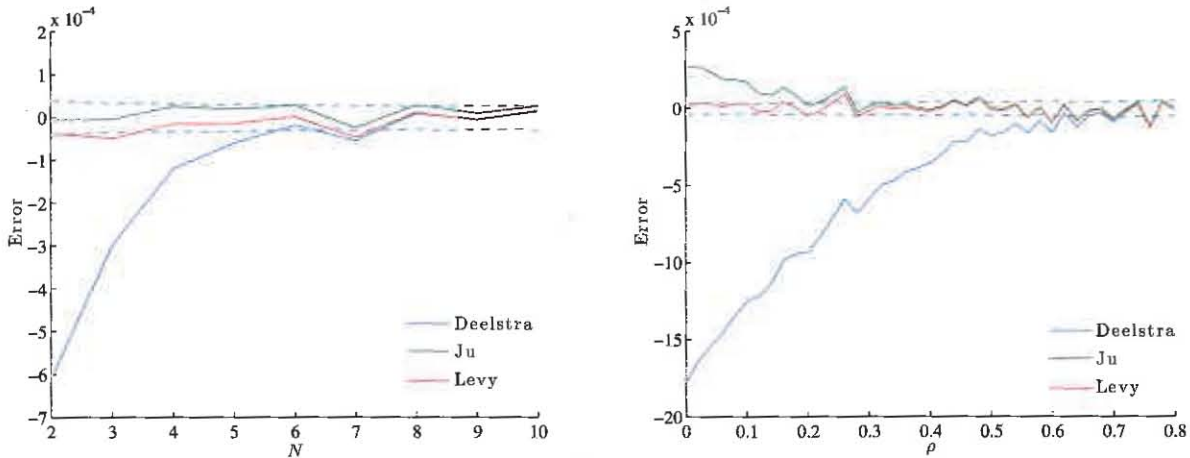


Figure 1: The sensitivity of the Levy, Ju and Deelstra approximations to N , the number of stocks, and ρ , the correlation of stocks in the basket.

Figure 1 shows the sensitivity of the approximations to the number of stocks in the basket and correlation. These two parameters seem to act in the same way - at high values of N and ρ all the methods work their way into the error bars and become very accurate. This is probably because a basket with a

large number of stocks is more predictable as unique risks get hedged out (in the language of CAPM), leaving the single factor of market volatility. This is easier to hedge and price, in any regime. Correlation is the same: with high correlation the stock paths track together and become harder to distinguish, leading to valuations which all the methods can handle with relative ease.

With low correlation and few stocks, Deelstra underprices quite severely compared to the other methods. This is not restricted to the conditioning variable used but was a feature for all the Deelstra formulations. This indicates that one should perhaps use other methods for baskets with small N or ρ .

5.2.2 Volatility and time to maturity

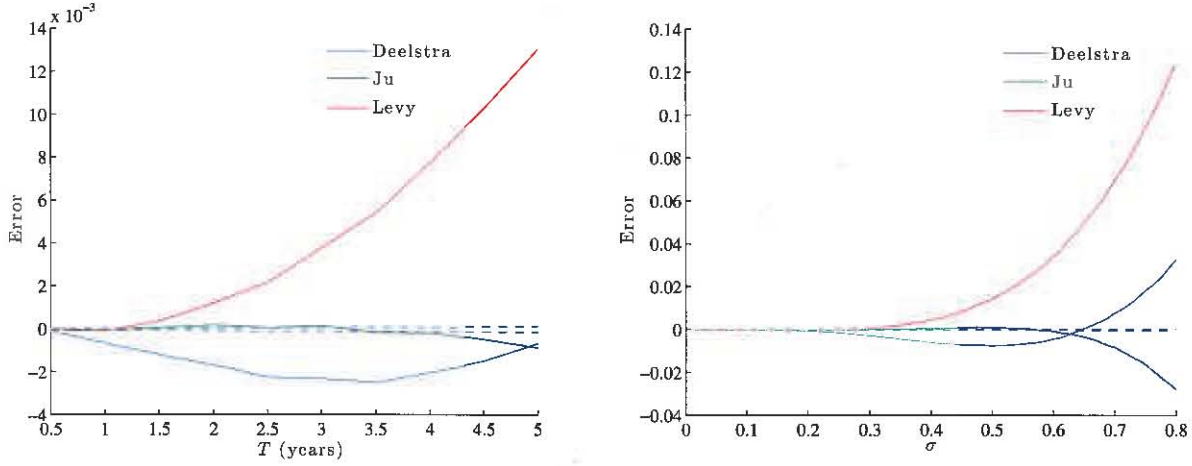


Figure 2: The sensitivity of the Levy, Ju and Deelstra approximations to σ , the volatility, and T , the time to maturity.

Figure 2 shows the sensitivity of the approximations to volatility and time to maturity. As stated above, σ and T act in the same way in the Black Scholes world of lognormal prices, as they appear in the same parts of all the formulas. The graphs confirm this.

What is immediately obvious is that the Levy approximation is problematic at high volatilities. This is in line with the results of Floor (2010), who found the same trend with the two moment method when pricing Asian options. Levy seems to do well in all other parameter regimes, but falls short here.

Another interesting feature is that the Deelstra errors are consistently on the downside as T increases. This distinct bias is different to the oscillating

errors which are more characteristic of the Ju approximation throughout the sensitivity analysis, as figures 3 and 4 confirm below. Oscillating errors are more desirable than biased errors of the same order.

5.2.3 Basket weights and the risk free rate

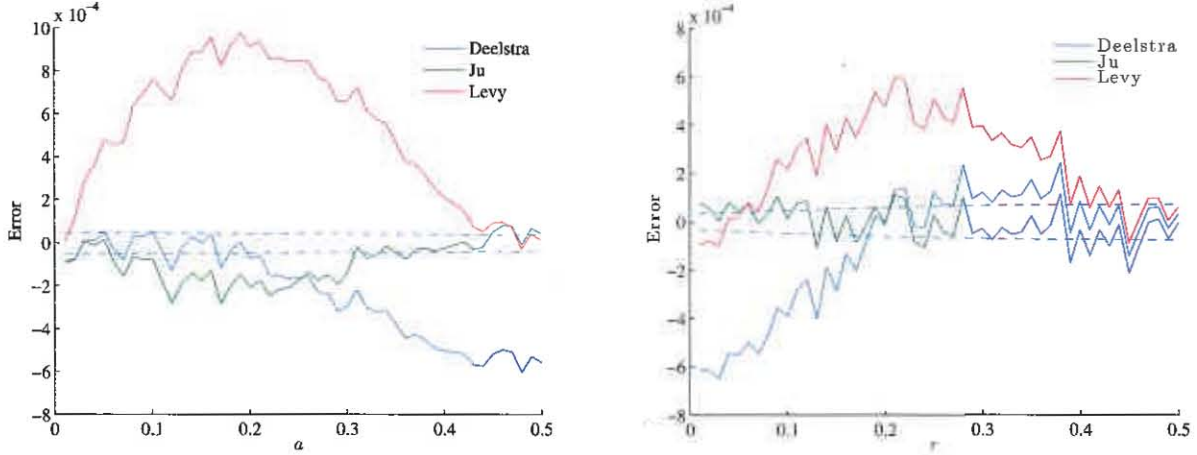


Figure 3: The sensitivity of the Levy, Ju and Deelstra approximations to a , the basket weights, and r , the risk free rate.

Figure 3 shows how basket weights and the risk free rate effect the accuracy of the three approximations. With these two parameter sets all the prices are consistently closer to the benchmark than in figures 1 or 2, which indicates that none of these parameter regimes cause serious difficulties for the approximations, as is the case with high volatility for example.

Nevertheless, we can see that Deelstra once again tends to underprice, while Levy overprices in these two instances. Ju is the most consistently inside the error bars¹¹.

5.2.4 Moneyness

Figure 4 shows the sensitivity of the approximations to changes in moneyness. As can be seen, a key result is that Deelstra again underprices the true price, especially for at-the-money options.

¹¹It should be noted that the jaggedness of the graphs is not a property of the approximations — they are an artifact of the Monte Carlo prices. Their standard error is of the same order, though less, than the approximation errors in these benevolent parameter regimes.

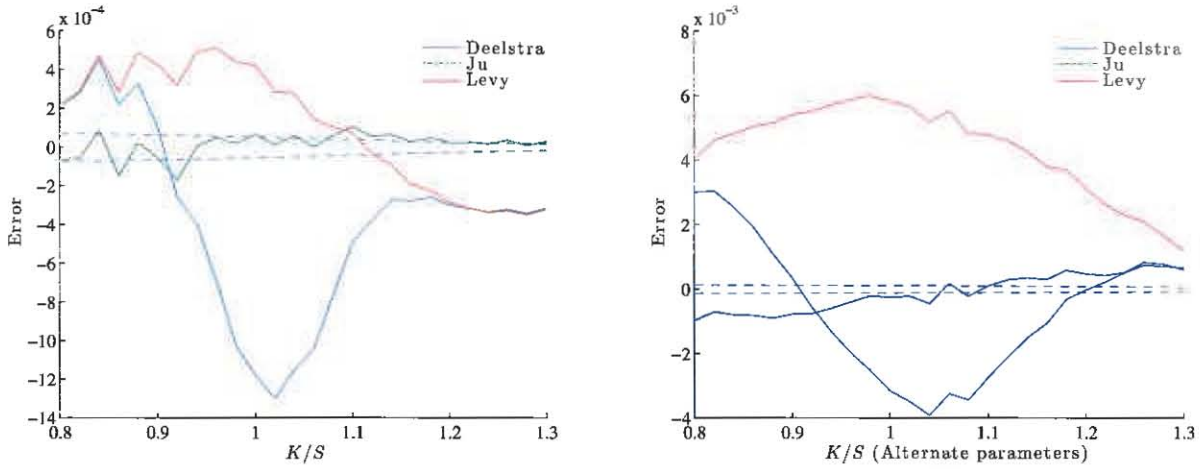


Figure 4: The sensitivity of the Levy, Ju and Deelstra approximations to K/S , the moneyness of the option. The alternate parameters are $T = 1.5$, $r = 0.15$, $\sigma = 0.3$ while the rest stay the same.

Another interesting feature is that Levy and Deelstra converge together as the option gets further into the money, but the convergence is not to the true price as one might expect.

This strange feature prompted the second graph in Figure 4, which also varies moneyness but for an adjusted set of parameters. In this case Ju and Deelstra converge to a false price, which indicates that this region should be treated with caution regardless of the approximation being used.

This second graph also illustrates that while a different parameter set leads to different individual errors, the salient features of the sensitivity analysis remain the same; for example Deelstra still underprices for at-the-money options. We can be more confident that the findings in section 5.2 are not specific to our base parameter set, but show trends that hold for a wider parameter range.

6 Conclusion

Basket options are tricky to price and sophisticated techniques are required to achieve accuracy. The three methods tested here by Levy, Deelstra and Ju all show a remarkable degree of accuracy: the errors are smaller than 10^{-2} in most cases.

A table of 24 parameter sets shows that by the root-mean-square and maximum-absolute-error measures, Levy is inferior to Deelstra and Ju, the two of which

perform comparably well.

Sensitivity analysis shows that the methods all struggle with high volatility, with biased errors growing as σ approaches 0.4 for Levy, and 0.7 for Deelstra and Ju. The same is true for long maturities.

Deelstra has weaknesses when pricing options with high unique risk, such as where correlation is low or there are few stocks in the basket.

Stock weights and the risk free rate don't show any specific trend in affecting the approximations, except that Deelstra continues to underprice and Levy shows signs of overpricing. This is also the case when K/S is varied.

A further key distinguishing factor in the performance of the three candidates is that while Ju has errors which tend to oscillate around the true price, Deelstra errors are more likely to be biased to underpricing, and Levy errors to overpricing. While this is not always the case, the oscillating character of Ju's errors are more desirable.

These results indicate that the Ju method for valuing basket options is the most-consistent, and should be the analytical approximation of choice.

6.1 Further research

While this paper has focused on the pricing of basket options with analytical approximations, the speed of these methods is perhaps even more useful when dynamically hedging such options. All of the three methods presented in this paper allow for the calculation of the greeks, and study into the performance of the approximations in this area would provide a more complete picture of their worth.

Another area of interest to be explored is the inclusion of term structures for risk free rates and volatility in the analytical approximations. If tractable, such modifications could lead to much more valuable results in practice.

Appendices

A Choosing the best Deelstra

As discussed in section 4.3, Deelstra et al. (2004) develop a total of 15 approximations in the course of their paper. Not all of them are equal, and it is important to choose the best one to represent their work.

For the analysis below we leave out upper bounds and use the \mathbb{S} method to represent the moment matching methods. This leaves us with six methods, which is further reduced to four, because the lower bound approximations price baskets identically when the the stocks are equally weighted with the same S_0 . When stocks are not equally weighted we use the FA2 conditioning variable for the lower bound, labelled LB. The three moment matching methods are labelled \mathbb{S}_{FA1} , \mathbb{S}_{FA2} and \mathbb{S}_{GA} respectively, where the subscripts indicate the conditioning variable.

To compare the four different Deelstras we will use the same two methods as in section 5: summary statistics and sensitivity analysis.

A.1 Summary statistics

Here we use the same parameter set as the table in section 5.1 to compare the methods. For the sake of brevity we include only the summary statistics, which are shown in table 2. These indicate that \mathbb{S}_{FA2} is the best among the Deelstra approximations.

Table 2: Summary statistics for various Deelstra approximations, using the same parameter set as in table 1.

	LB	\mathbb{S}_{FA1}	\mathbb{S}_{FA2}	\mathbb{S}_{GA}
RMSE	0.1410	0.0152	0.0115	0.1112
MAE	0.5301	0.0617	0.0439	0.5325

RMSE is root mean square error and MAE is maximum absolute error.

A.2 Sensitivity analysis

For further diagnostics we do sensitivity analysis, as in section 5.2 above. We cannot include all the graphs but two are enough to portray the essence of the results. Figure 5 shows how the accuracy of the four Deelstra methods vary with T and a , the time to maturity and basket weights respectively.

The first graph, pertaining to T , is included to immediately demonstrate that the lower bound method and the GA conditioning variable for moment matching are inadequate and should be ignored. Finding a preference between the FA1 and FA2 conditioning variables is not as straightforward, but most of the sensitivity analysis graphs, of which the one with basket weights is shown here, indicate that FA2 is slightly better.

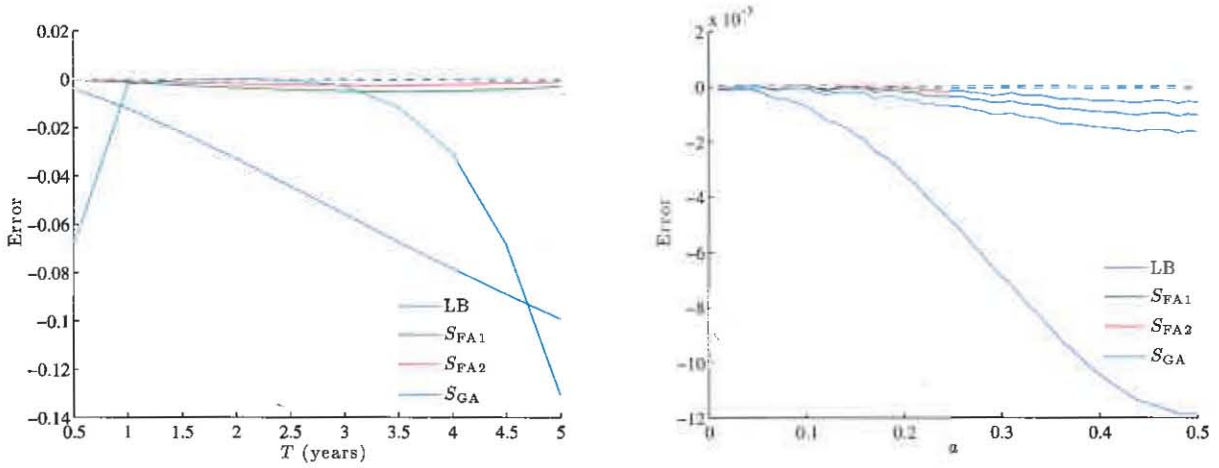


Figure 5: The sensitivity of four Deelstra approximations to T , the time to maturity, and a , the basket weights.

From these findings we conclude that the best Deelstra method is moment matching with FA2 as the conditioning variable.

B Some code

I have included below one of the Matlab functions I coded in the course of this project; namely the Deelstra moment matching method of type S. The conditioning variable is an input to the function and is dealt with in subsidiary functions, not hardcoded here.

The functions called in this piece of code are `getR`, `dlam`, `varlam`, `quadOwn`, `roundOwn` and `cumNormOwn`. The first three return values pertaining to the conditioning variable used, while the last three are personalised methods that already exist in Matlab, but with features I found easier to write my own code for. `quadOwn` handles quadrature of Matlab function-handles and uses Simpson's rule. The loop from line 49 to 57, while not at all optimal, achieves accuracy to

the fourth decimal, which proved adequate. cumNormOwn is the standard normal distribution and was written because I didn't have the Matlab statistics toolbox at the time.

This method is not instantaneous (or nearly instantaneous) as an analytical approximation should be, partly because of the quadrature required but mainly because optimisation was not the objective here. Rather, much effort was made to make sure of absolute accuracy. Tests comparing these prices (and the other Deelstra methods) to the values in the original paper found no discrepancies.

```

1 function price=approxDeelMoments(S0,a,K,sigma,T,corrs,r,type)
2
3 R=getR(S0,a,r,sigma,T,corrs,type);
4
5 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
6 %exact part
7 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
8 dlambd=dlam(S0,a,r,sigma,T,corrs,K,type);
9 varlambd=varlam(S0,a,r,sigma,T,corrs,K,type);
10
11 dlambdStar=(dlambd-0)/sqrt(varlambd);
12
13 exact=sum(a.*S0.*cumNormOwn(R.*sigma.*sqrt(T)-dlambdStar))-...
14     exp(-r.*T).*K.*cumNormOwn(-dlambdStar);
15
16 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
17 %inexact part
18 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
19 expectS=@(l) sum(a.*S0.*exp((r-sigma.*sigma.*R.*R./2).*T+sigma...
20     .*R.*sqrt(T).*((1-0)./sqrt(varlambd))));
21 expectSS=@(l) 0.*1+0;
22 for i = 1:length(S0) %doing this the long way
23     for j=1:length(S0)
24         sigmai=sqrt(sigma(i).*sigma(i)+sigma(j).*sigma(j)+2.*...
25             sigma(i).*sigma(j).*corrs(i,j));
26         rij=(sigma(i).*R(i)+sigma(j).*R(j))./sigmai;
27         temp=@(l) ...
28             a(i).*a(j).*S0(i).*S0(j).*exp((2.*r-((sigma(i)...
29                 .*sigma(i)+sigma(j).*sigma(j))./2)).*T+rij.*sigmai...
30                 .*sigmai./2);
31         expectSS=@(l) expectSS(l)+temp(l);

```

```

32     end
33 end
34
35 sigmafunc=@(l) sqrt(log(expectSS(l))-2.*log(expectS(l)));
36 d1=@(l) (0.5.*log(expectSS(l))-log(K))./sigmafunc(l);
37 d2=@(l) d1(l)-sigmafunc(l);
38
39 cumNormOwnd1=@(l) cumNormOwn(d1(l));
40 cumNormOwnd2=@(l) cumNormOwn(d2(l));
41
42 pdf=@(l) ...
    1./sqrt(2.*pi.*varlambda).*exp(-(l-0).^2./2./varlambda);
43
44 integrand=@(l) ...
    (expectS(l).*cumNormOwnd1(l)-K.*cumNormOwnd2(l)).*pdf(l);
45
46 acc=false;
47 intervals=5e2;
48 inexact1=exp(-r*T)*quadOwn(integrand,-100,dlambda,intervals);
49 while acc==false
50     inexact2=exp(-r*T)*quadOwn(integrand,-100,dlambda,intervals*2);
51     if roundOwn(inexact1,4)==roundOwn(inexact2,4)
52         acc=true;
53     else
54         intervals=intervals*2;
55         inexact1=inexact2;
56     end
57 end
58 inexact=inexact2;
59
60 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
61 %final
62 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
63 price=exact+inexact;

```

7 References

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